

Modelling non-dust fluids in cosmology

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Often in cosmology Newtonian physics is used to model the dynamics of matter inhomogeneities. For pressureless dark matter, or ‘dust’, this approach gives the correct results, but for scenarios in which the fluid has pressure this is no longer the case. In this article, we explicitly highlight the relationship between the variables in Newtonian and cosmological perturbation theory, showing exact equivalence for pressureless matter, and giving the relativistic corrections for matter with pressure. As an example, we focus on the scalar field dark matter model, which has recently gained popularity but which has non-zero pressure perturbations. We discuss some problems which may arise when modelling this theory with numerical simulations, and when using CMB Boltzmann codes.

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I. INTRODUCTION

Current observations indicate that the universe in which we live is almost homogeneous and isotropic. However, it is also known that small initial departures from homogeneity and isotropy give rise to the structures we observe today. Thus, while the universe is well approximated on large scales by a homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker (FLRW) space-time, the existence of large scale structure and inhomogeneities in the Cosmic Microwave Background (CMB) tell us that this is just an approximation [1].

In order to model inhomogeneities we utilise a technique that is well established in many branches of physics and applied mathematics, namely, we take an approximate solution and add small perturbations. In cosmology, this process is called cosmological perturbation theory, and requires the addition of small inhomogeneous perturbations on top of the FLRW background, such that the system still solves Einstein’s field equations (e.g. Refs. [2–4]). However, when considering dynamics on sufficiently small scales, and for fluids which are pressureless, it is sufficient to use Newtonian physics [5]. Inhomogeneous perturbations of Newtonian cosmology have been studied for many years and these are the equations that are used when performing large numerical simulations of galaxy formation [6]. However, as simulations get more sophisticated and the observations more accurate, it is important to consider the contribution of those components which exhibit pressure perturbations. For example, to fully account for the effects of inhomogeneous scalar

fields as dark energy (e.g. [7]) or dark matter [8–11] in the formation of structures, one should employ more general equations with the input of general relativity.

In this paper we re-visit the question of how to relate Newtonian and cosmological perturbation theories. We show that for pressureless systems the equations governing cosmological perturbations of an FLRW space-time reduce to the equivalent Newtonian equations on using gauge invariant variables defined in different gauges, namely the metric potential in the longitudinal gauge, and the density contrast in the comoving gauge [12]. Drawing on this equivalence, we then investigate the situation for fluids with pressure. We find that one can write the Poisson equation in the usual way, but that the continuity and Euler equations then differ, depending upon the equation of state parameter and the adiabatic sound speed.

We then go on to study how to relate the two perturbation theories in a scalar field model. We find that, again, the Poisson equation is identical to the Newtonian case, but that the Euler and continuity equations differ, depending now on the equation of state parameter and the effective speed of propagation of perturbations through the system. Finally we discuss the Jeans scale in the scalar field dark matter models, where the background equation of state parameter is zero. We find that the scales depend upon c_{ph}^2 , the phase speed, or speed of propagation of perturbations. We close, in Section IV, with a brief discussion.

II. MODELLING INHOMOGENEITIES

A. Newtonian perturbations

Let us first study the theory of perturbations in Newtonian physics. We consider inhomogeneous perturbations

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about a homogeneous background, and so the energy density ρ is

$$\rho(\vec{x}, t) = \bar{\rho}(t) \left(1 + \delta(\vec{x}, t) \right), \quad (2.1)$$

where $\bar{\rho}(t)$ is the homogeneous background energy density and $\delta(\vec{x}, t)$ is the inhomogeneous density contrast. On introducing the inhomogeneous Newtonian potential, $\Phi_N(\vec{x}, t)$ and fluid velocity $\vec{v}(\vec{x}, t)$, the linearised conservation and Euler equations are, respectively, [5, 13]

$$\dot{\delta} + \frac{1}{a} \vec{\nabla} \cdot \vec{v} = 0, \quad (2.2)$$

$$\dot{\vec{v}} + H\vec{v} = -\frac{1}{a} \vec{\nabla} \Phi_N - \frac{1}{a\bar{\rho}} \vec{\nabla} \delta P, \quad (2.3)$$

where $H = \dot{a}/a$ is the Hubble parameter, a dot denotes a derivative with respect to coordinate time and δP denotes the pressure perturbation. The Newtonian potential and the density contrast are then related through the Poisson equation

$$\nabla^2 \Phi_N = 4\pi G a^2 \bar{\rho} \delta, \quad (2.4)$$

where the Laplacian is defined as $\nabla^2 = \vec{\nabla} \cdot \vec{\nabla}$. On utilising the relationship between the energy density perturbation and the pressure perturbation for a barotropic fluid, $\delta P = c_s^2 \delta \rho$, we can combine the fluid equations into a second order equation

$$\frac{\partial^2 \delta}{\partial t^2} + 2H \frac{\partial \delta}{\partial t} = 4\pi G \bar{\rho} \delta + c_s^2 \nabla^2 \delta. \quad (2.5)$$

B. Cosmological perturbations

While the theory of Newtonian perturbations is sufficient for modelling small scale physics involving only pressureless dark matter particles, the dynamics of the universe are governed by general relativity. Therefore, we must consider relativistic perturbation theory. Since Einstein's theory relates the geometry of the universe to its matter content, we must consider perturbations of both the matter and the FLRW spacetime metric.

The most general, linear scalar perturbations to the FLRW metric are [2, 3, 14]

$$ds^2 = a^2(\eta) \left[- (1 + 2\phi) d\eta^2 + 2B_{,i} dx^i d\eta + \{ (1 - 2\psi) \delta_{ij} + E_{,ij} \} dx^i dx^j \right], \quad (2.6)$$

where we now use conformal time η , related to coordinate time t through $dt = a d\eta$. A unique problem which arises in the relativistic theory is the problem of gauge dependence. Since general relativity is covariant, and splitting the spacetime into a background and a perturbation is not a covariant process, we introduce extra spurious coordinate dependence (see, e.g., Refs. [4, 15, 16]). This can be resolved in a systematic manner, as was first shown

by Bardeen [12], by considering gauge-invariant variables – quantities that do not change under a gauge transformation. A popular choice of variables amount to setting E and B to zero, resulting in the FLRW metric in the so-called longitudinal or Newtonian gauge, with the gauge invariant variables Φ and Ψ

$$ds^2 = a^2(\eta) \left[- (1 + 2\Phi) d\eta^2 + (1 - 2\Psi) \delta_{ij} dx^i dx^j \right]. \quad (2.7)$$

The conservation and Euler equations are then reduced from the form without fixing the gauge

$$\delta\rho' + 3\mathcal{H}(\delta\rho + \delta P) - 3(\bar{\rho} + \bar{P})\psi' + (\bar{\rho} + \bar{P})\nabla^2(v + E') = 0 \quad (2.8)$$

$$V' + \left(1 - 3\frac{\bar{P}'}{\bar{\rho}'} \right) \mathcal{H}V + \phi + \frac{1}{\bar{\rho} + \bar{P}} \left(\delta P + \frac{2}{3} \nabla^2 \Pi \right) = 0 \quad (2.9)$$

to the following expressions, where we neglect anisotropic stresses, Π ,

$$\delta\rho'_\ell + (\bar{\rho} + \bar{P})(\nabla^2 v_\ell - 3\Psi') = -3\mathcal{H}(\delta\rho_\ell + \delta P_\ell), \quad (2.10)$$

$$[(\bar{\rho} + \bar{P})v_\ell]' + 4\mathcal{H}(\bar{\rho} + \bar{P})v_\ell + (\bar{\rho} + \bar{P})\Phi + \delta P_\ell = 0. \quad (2.11)$$

Here the subscript ℓ denotes matter variables in the longitudinal gauge and v is the velocity potential, i.e. $v^i = \nabla^i v$ with v^i the fluid three velocity, and $V \equiv v + B$.

We can then specialise to a barotropic fluid with equation of state $\bar{P} = w(\eta)\bar{\rho}$, and whose pressure perturbation can be related to the energy density perturbation through

$$\delta P = c_s^2 \delta \rho, \quad (2.12)$$

where $c_s^2 = \bar{P}'/\bar{\rho}'$ is the adiabatic sound speed¹. Thus, for this system, Eqs. (2.10) and (2.11) become

$$\delta'_\ell + (1 + w)(\nabla^2 v_\ell - 3\Psi') = 3\mathcal{H}(w - c_s^2)\delta_\ell, \quad (2.14)$$

$$v'_\ell + \mathcal{H}(1 - 3c_s^2)v_\ell + \Phi + \frac{c_s^2}{1 + w}\delta_\ell = 0. \quad (2.15)$$

The Einstein equations then give that $\Phi = \Psi$ (in the case of zero anisotropic stress, as is true for any perfect fluid), and the Poisson equation is

$$\nabla^2 \Psi = 4\pi G a^2 \bar{\rho} [\delta_\ell - 3\mathcal{H}(1 + w)v_\ell]. \quad (2.16)$$

¹ The adiabatic sound speed, equation of state parameter and its derivative are related through

$$\frac{w'}{1 + w} = -3\mathcal{H}(c_s^2 - w). \quad (2.13)$$

For the case where w is constant, these relationships guarantee that $w = c_s^2$.

In order to draw an equivalence between this and the Newtonian Poisson equation, we can set $\Psi = \Phi_N$. Furthermore, on considering that the energy density perturbation transforms under the gauge transformation $x^\mu \rightarrow \widetilde{x}^\mu \equiv x^\mu + \delta\eta$ as

$$\widetilde{\delta\rho} = \delta\rho + \bar{\rho}'\delta\eta, \quad (2.17)$$

we get for the comoving density contrast, in terms of the longitudinal density contrast and the velocity perturbation [14],

$$\delta_c = \delta_\ell - 3\mathcal{H}(1+w)v_\ell. \quad (2.18)$$

Then we obtain the Poisson equation [12, 17]

$$\nabla^2\Phi_N = 4\pi G a^2 \bar{\rho} \delta_c, \quad (2.19)$$

which is equivalent to Eq. (2.4).

C. Correspondences

In this Section we relate the relativistic equations to the Newtonian equations. We take heed from the Poisson equation which, as stated above, relates the density contrast in the comoving gauge to the metric potential in the longitudinal gauge:

$$\nabla^2\Psi = 4\pi G a^2 \bar{\rho} \delta_c. \quad (2.20)$$

In order to obtain an equivalence for the set of equations, we must first ensure that we are consistent with the density contrast that we use. With the aid of the background Friedmann equations and one of Einstein's perturbed equations,

$$\Psi' = -\mathcal{H}\Psi - 4\pi G a^2(1+w)\bar{\rho}v_\ell, \quad (2.21)$$

we can rewrite Eq. (2.10) as

$$\begin{aligned} \delta'_\ell + 3\mathcal{H}(c_s^2 - w)\delta_\ell + (1+w)(3\mathcal{H}\Psi + \nabla^2 v_\ell) \\ + \frac{9}{2}\mathcal{H}^2(1+w)^2 v_\ell = 0. \end{aligned} \quad (2.22)$$

Using now the relationship between δ_ℓ and δ_c , Eq. (2.18), and its derivatives, the continuity equation can be written as

$$\delta'_c - 3\mathcal{H}w\delta_c + (1+w)\nabla^2 v_\ell = 0. \quad (2.23)$$

Furthermore, the Euler equation, (2.11), is

$$v'_\ell + \mathcal{H}v_\ell = -\Psi - \frac{c_s^2}{1+w}\delta_c. \quad (2.24)$$

From this we can see that, in the case of a pressureless dust for which $w = c_s^2 = 0$, the evolution equations reduce to

$$\delta'_c + \nabla^2 v_\ell = 0, \quad (2.25)$$

$$v'_\ell + \mathcal{H}v_\ell + \Psi = 0, \quad (2.26)$$

which can be recast as a second order differential equation as

$$\delta_c'' + \mathcal{H}\delta_c = 4\pi G a^2 \bar{\rho} \delta_c, \quad (2.27)$$

which governs the evolution of matter density perturbations. Here the second term is a suppression of the perturbations with the expansion of the universe, and the term on the right hand side sources the growth of perturbations due to the gravitational instability. This is the equivalent form to the Newtonian perturbation theory (as expected for a non-relativistic species), as obtained in Eq. (2.5). This equivalence was discussed in greater detail in Ref. [18].

However, should one wish to consider fluids other than dust, with non-zero pressure perturbations, then the relativistic equations must be used. This is especially important for, say, hot dark matter, or for a system containing dark energy perturbations. In the following, we will consider the case of scalar field dark matter models.

III. SCALAR FIELD DARK MATTER

A model of dark matter which is receiving increased attention is the scalar field dark matter model (SFDM), specifically in the form of a Bose-Einstein condensate [19, 20], the theory is robust enough to study the numerical problem of structure formation (recent attempts are [21–23]). The basic model features a canonical scalar field oscillating at the bottom of its potential with a period much smaller than the Hubble time and any other dynamical times. In consequence the SFDM can be regarded as a fluid with zero effective pressure. However, in contrast to cold dark matter (CDM), this component exhibits pressure perturbations.

It is well known that a scalar field system cannot be modelled as a barotropic fluid (except on super-horizon scales [24]), and in fact a consistent fluid equivalence must consider the more general perfect fluid [25]. Indeed, as elucidated in Ref. [26], in order to interpret a scalar field as a fluid, the equation of state parameter, w and adiabatic sound speed c_s^2 differ. Furthermore, the speed with which pressure perturbations propagate is described by the effective sound speed, or phase speed, c_{ph}^2 defined, in the fluid rest frame, as

$$c_{\text{ph}}^2 = \left. \frac{\delta P}{\delta \rho} \right|_{\text{rf}}. \quad (3.1)$$

For a scalar field with a canonical kinetic term, this is equal to unity. However, for a non-canonical scalar field with pressure and energy density depending upon both the field, φ , and its kinetic term, $X \equiv -\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi$,

this can differ from one and is given by [24, 27]²

$$c_{\text{ph}}^2 = \frac{P_{,X}}{\rho_{,X}}. \quad (3.2)$$

For SFDM, the pressure perturbation is no longer proportional to the energy density perturbation. Instead the relationship between pressure perturbations and energy density perturbations is

$$\delta P = c_s^2 \delta \rho + (c_{\text{ph}}^2 - c_s^2) [\delta \rho + \bar{\rho}'(v + B)]. \quad (3.3)$$

The second term on the right hand side is often referred to as the non-adiabatic pressure perturbation δP_{nad} . In this case, Eqs. (2.10) and (2.11) become

$$\begin{aligned} \delta'_\ell + (1+w)(\nabla^2 v_\ell - 3\Psi') \\ = 3\mathcal{H}(w - c_{\text{ph}}^2)\delta_\ell + 9\mathcal{H}^2(1+w)(c_{\text{ph}}^2 - c_s^2)v_\ell, \end{aligned} \quad (3.4)$$

$$\begin{aligned} v'_\ell + \mathcal{H}(1-3w)v_\ell + \Phi + \frac{c_{\text{ph}}^2}{1+w}\delta_\ell + \frac{w'}{1+w}v_\ell \\ = 3\mathcal{H}(c_{\text{ph}}^2 - c_s^2)v_\ell. \end{aligned} \quad (3.5)$$

As shown previously, in order to write the equations in the variables employed in simulations, we introduce the density contrast in the comoving gauge. Following a similar procedure, using the background and perturbed Einstein equations, we obtain

$$\begin{aligned} \delta'_\ell + 3\mathcal{H}(c_{\text{ph}}^2 - w)\delta_\ell + 3\mathcal{H}^2(1+w)\Psi + (1+w)\nabla^2 v_\ell \\ + 9\mathcal{H}^2(1+w)(c_s^2 - c_{\text{ph}}^2 + \frac{1+w}{2})v_\ell = 0, \end{aligned} \quad (3.6)$$

which can then be written as

$$\delta'_c - 3\mathcal{H}w\delta_c + (1+w)\nabla^2 v_\ell = 0. \quad (3.7)$$

Interestingly, this is identical to the equation for the barotropic fluid presented earlier. However, the Euler equation, Eq. (2.24), does not take the same form, and can be written in terms of the comoving density contrast as

$$v'_\ell + \mathcal{H}v_\ell = -\Psi - \frac{c_{\text{ph}}^2}{1+w}\delta_c. \quad (3.8)$$

The combination of these two equations yields a Klein-Gordon equation – a generalisation of Eq. (2.27) for δ with an extra Laplacian term,

$$\begin{aligned} \delta''_c - c_{\text{ph}}^2 \nabla^2 \delta_c + \mathcal{H}(3w-1)\delta'_c \\ - 3\mathcal{H}^2 \left(\frac{1}{2} - 3w + \frac{9}{2}w^2 \right) \delta_c = 0. \end{aligned} \quad (3.9)$$

From Eq. (3.9) one can determine the instability scale for density perturbations. For the most general, scalar field case we obtain the Jeans wavenumber

$$k_J^2 = \frac{3}{2} \frac{\mathcal{H}^2}{c_{\text{ph}}^2} \left(3w^2 - \frac{5}{2}w + 1 \right). \quad (3.10)$$

This scale has been associated to a power spectrum cut-off, and is a characteristic feature of non-dust components. In the case of the scalar field acting as a dark matter component, for which $w = 0$, the Jeans' wavelength reduces to

$$\lambda_J = c_{\text{ph}} \sqrt{\frac{\pi}{G\rho}}. \quad (3.11)$$

This is equivalent to the field calculation (see, e.g., Ref. [28]) but with the subtle difference that the effective sound speed, c_{ph} , takes the place of the adiabatic sound speed c_s (see also [29]). This corresponds to the fact that the speed of propagation of perturbations in a scalar field system is given by c_{ph}^2 . The effective sound speed for each canonical SFDM model is equivalent and equal to 1, though for non-canonical models this differs (see e.g. [30]). The specific scales for each model, as well as the growth of perturbations will be treated elsewhere.

These simple results show the importance of considering the system with no approximations and argue for the use of the system of equations (3.7) and (3.8) in the forthcoming simulations of structure formation in the SFDM model.

Another important comment is that care must be taken when studying the SFDM model using CMB codes. Many of the popular Boltzmann codes are written in the synchronous gauge, a gauge specified by demanding that $\phi = 0 = B$. However, as has been known for some time, these conditions alone do not fix the gauge. That is, the synchronous conditions alone are not a proper gauge choice, and one needs an additional condition. The condition usually chosen, and which is employed by, e.g., CAMB [31], is to set the velocity perturbation of the cold dark matter to zero.

This can be done since the dark matter is assumed to have zero pressure perturbations [32]. But this is not possible for models with pressure perturbations, since imposing this condition will result in inconsistencies in the theory. Consequently, in order to use CMB codes to study the SFDM model, one must either include an extra CDM component to allow for this condition – which arguably reduces the value of the theory – or use a code that is not written in the synchronous gauge.

IV. DISCUSSION

In this paper we have revisited the issue of relating Newtonian perturbation theory to relativistic perturbation theory in cosmology. Having reviewed both perturbation theories, we then explicitly showed how one relates

² Note that the equivalence between Eqs. (3.1) and (3.2) is only true for a single scalar field, which is the case we study in this paper.

the two for the case of a pressureless fluid, such as dark matter. As is quite well documented in the literature on cosmological perturbation theory, in order to relate the two approaches one must use the Newtonian gauge with the comoving energy density perturbation.

We then considered the extension of this for fluids with pressure, taking our lead from the dark matter case, and using the Newtonian gauge with the comoving energy density perturbation, which results in the Poisson equation being identical to the Newtonian case. In this case the continuity equation differs, depending upon the equation of state parameter, and the Euler equation depends upon c_s^2 .

The major application that we explored in Section III regards scalar field dark matter. In this case, the dark matter species is no longer pressureless, but is instead a scalar field with canonical kinetic term. In addition to the adiabatic sound speed and equation of state parameter, which are introduced for a general perfect fluid, a scalar field also has a speed of propagation of perturbations, which we dubbed the phase, or effective sound, speed. The evolution equations for this scalar field fluid in the longitudinal gauge depend on all the parameters. However, on writing the equations in terms of the comoving energy density perturbation, the Poisson and continuity equations reduce to those of perfect fluid form. The Euler equation, on the other hand, still depends upon all the parameters. Thus we have shown that, when studying the scalar field dark matter model and treating it

as a fluid, one cannot simply use the pressureless cold dark matter equations without potentially finding erroneous results. Instead, one must use the equations given in Section III.

This non-equivalence of the relativistic theory to the Newtonian theory is also important in systems containing more than one fluid. For example, as shown in Ref. [33], in a system containing normal CDM and a dark energy component, Eq. (2.27) is no longer satisfied, since one must take into account the dark energy perturbations. This may also be important when attempting to model a system with dark matter and another species.

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